

# Spheres of small diameter with long sweep-outs

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## Abstract

We prove the absence of a universal diameter bound on lengths of curves in a sweep-out of a Riemannian 2-sphere. If such bound existed it would yield a simple proof of existence of short geodesic segments and closed geodesics on a sphere of small diameter.

## 1 Introduction

By a sweep-out of a Riemannian 2-sphere  $M = (S^2, g)$  we mean a non-contractible loop  $\gamma_t$  in  $(\Lambda M, \Lambda^0 M)$ , where  $\Lambda M$  denotes the space of free loops on  $M$  and  $\Lambda^0 M$  denotes the space of constant loops. In other words,  $\gamma_t$  is a 1-parameter family of closed curves, starting and ending at a point and inducing a non-zero degree map  $f : S^2 \rightarrow M$ . If there exists a sweep-out by loops of length  $\leq L$  then the standard minimax argument implies that there exists a non-trivial closed geodesic on  $M$  of length  $\leq L$  ([B]).

Moreover, if there exists a sweep-out of  $M$  by loops with 2 fixed points and of length  $\leq L$  then for any two points  $a, b \in M$  there exists  $k$  distinct geodesic segments from  $a$  to  $b$  of length  $\leq 2kL + 2\text{diam}(M)$  (cf. [NR1]). If we could take  $L \leq C\text{diam}(M)$  we would obtain a linear in  $k$  bound on the length of  $k$ -th shortest geodesic segment in terms of the diameter.

Yet A.Nabutovsky and R.Rotman observe (see [N], [NR1], [NR2]) that example of S.Frankel and M.Katz [FK] (see below) suggests that for any  $C$  there exists a sphere with no sweep-out obeying this inequality. The author learnt from Regina Rotman that a similar conjecture was independently made by S.Sabourau. In this note we prove this conjecture.

## 2 Main result

Let  $\gamma : I \times S^1 \rightarrow M$  be a 1-parameter family of free loops (we write  $\gamma_t(s)$  for  $\gamma(t, s)$ ), such that  $\gamma_0(s)$  and  $\gamma_1(s)$  are constant loops.  $\gamma_t$  induces a map  $f : S^2 \rightarrow M$ , such that  $\gamma(t, s) = f \circ p(t, s)$ , where  $p : I \times S^1 \rightarrow S^2$  is the suspension map that collapses  $\{0\} \times S^1$  to the South pole and  $\{1\} \times S^1$  to the North pole of  $S^2$ . If  $\deg(f) \neq 0$  we call  $\gamma_t$  a sweep-out.

**Theorem 1.** *For any  $C > 0$  there exists a Riemannian 2-sphere  $M$  of diameter  $\leq 1$ , such that for any sweep-out  $\gamma_t$  of  $M$  there is a loop  $\gamma_{t_0}$  of length  $\geq C$ .*

F.Balacheff and S.Sabourau in [BS] defined a *diastole* of  $M$  as

$$\text{dias}(M) := \inf_{(\gamma_t)} \sup_{0 \leq t \leq 1} \text{length}(\gamma_t)$$

where  $(\gamma_t)$  runs over the families of loops inducing  $f : S^2 \rightarrow M$  of degree  $\pm 1$ . In [S, Remark 4.10] S.Sabourau constructs a sequence  $M_n$  of Riemannian two-spheres such that

$$\lim_{n \rightarrow \infty} \frac{\sqrt{\text{Area}(M_n)}}{\text{dias}(M_n)} = 0.$$

Theorem 1 implies the analogous result with  $\text{diam}(M_n)$  in place of  $\sqrt{\text{Area}(M_n)}$ .

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F.Balacheff and S.Sabourau prove ([BS]) that if 1-parameter families of loops in the definition of the diastole are replaced with 1-parameter families of one-cycles then every Riemannian 2-sphere satisfies  $dias(M) \leq C\sqrt{Area(M)}$  for a universal constant  $C$ .

*Proof.* We use the example of S.Frankel and M.Katz [FK]. For any natural number  $N$  they embed a binary tree  $T$  of height  $N$  in a 2-dimensional disc  $D$  and define a Riemannian metric on  $D$ , such that the distance between any two non-adjacent edges of  $T$   $dist(e_i, e_j) \geq 1/2$ , but the diameter  $diam(D) \leq 1$ . They prove that for every homotopy of closed curves  $\gamma_t$  with  $\gamma_0 = \partial D$  and  $\gamma_1 = \{*\}$  there is an intermediate curve  $\gamma_{t_0}$  that intersects at least  $O(N/\log N)$  edges of  $T$  and hence must be at least  $O(N/\log N)$  long.

Let  $M = (S^2, g)$  be a sphere of diameter less than 1 containing the disc of Frankel and Katz with an embedded binary tree  $T$  of height  $N$ . Consider a sweep-out  $\gamma_t$  and let  $f : S^2 \rightarrow M$  be the induced map from the suspension of  $S^1$  to  $M$  ( $\gamma_t(s) = f \circ p(t, s)$ ).

Suppose that at time  $t_0 \in [0, 1]$   $\gamma_{t_0}$  does not pass through a vertex  $v \in T$ . Let  $E$  denote a connected component of  $M \setminus \gamma_{t_0}$  that contains  $v$ . Identify all points of  $M \setminus E$  to a new point  $x$  to obtain a quotient space  $E \cup \{x\}$  homeomorphic to  $S^2$ . The loop  $p(t_0, s)$  divides  $S^2$  into two connected components, the south component  $S_{t_0} = \{p(t, s) : t < t_0\}$  and the north component  $N_{t_0} = \{p(t, s) : t > t_0\}$ . Under the composition of  $f$  with the quotient map  $q_1 : M \rightarrow E \cup \{x\}$  the loop  $p(t_0, s)$  is mapped to the point  $x$ . If we collapse all points of  $p(t_0, s)$  to a point  $y$  we obtain a map  $f_{t_0, v} : S_{t_0} \cup \{y\} \cup N_{t_0} \rightarrow E \cup \{q\}$ , where the quotient space  $S_{t_0} \cup \{y\} \cup N_{t_0}$  is homeomorphic to the wedge sum  $S^2 \vee S^2$ . We have the following commutative diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{f} & M \\ q_2 \downarrow & & \downarrow q_1 \\ S_{t_0} \cup \{y\} \cup N_{t_0} & \xrightarrow{f_{t_0, v}} & E \cup \{x\} \end{array}$$

Let  $f_{t_0, v}^S$  denote the restriction of  $f_{t_0, v}$  to the south sphere  $S_{t_0} \cup \{y\}$  and  $f_{t_0, v}^N$  denote the restriction to the north sphere  $N_{t_0} \cup \{y\}$ . Observe that from the induced commutative diagram for the second homology groups we have  $deg(f) = deg(f_{t_0, v}^S) + deg(f_{t_0, v}^N)$ . Indeed, the map  $q_{2\#} : H_2(S^2) \rightarrow H_2(S^2 \vee S^2)$  sends a generator 1 to an element  $(1, 1) \in \mathbb{Z} \times \mathbb{Z}$  and  $(f_{t_0, v})_{\#}(a, b) = (f_{t_0, v}^S)_{\#}(a) + (f_{t_0, v}^N)_{\#}(b)$ , while  $q_{1\#}$  is an isomorphism.

Let  $A \subset [0, 1]$  denote the set of all  $t$  such that  $\gamma_t$  does not pass through  $v$ . We define a function  $d_v : A \rightarrow \mathbb{N}$  (degree of  $v$  at time  $t$ ) by

$$d_v(t) = deg(f_{t, v}^N) \tag{1}$$

We need two simple facts about  $d_v(t)$ :

Observation 1. If  $\gamma_t$  does not intersect  $v$  for  $t \in [t_1, t_2]$  then  $d_v(t)$  is constant on  $[t_1, t_2]$

*Proof:* Choose a small disc  $D$  around  $v$ , s.t.  $D \cap f(p([t_1, t_2]))$  is empty and define a quotient map  $q$  collapsing  $M \setminus D$  to a point. For each  $t \in [t_1, t_2]$  we can define a map between spheres  $q' : N_t \cup \{a\} \rightarrow N_{t_2} \cup \{b\}$  sending  $N_t \setminus N_{t_2}$  to  $b$ . Then  $q \circ f_{t, v}^N = q \circ f_{t_2, v}^N \circ q'$ . As  $deg(q) = deg(q') = 1$  we have  $deg(f_{t, v}^N) = deg(f_{t_2, v}^N)$

Observation 2. Let  $v_1, v_2$  be two vertices of  $T$  connected by an edge  $e$ . If  $\gamma_t$  does not intersect  $e$  then  $d_{v_1}(t) = d_{v_2}(t)$

*Proof:* Since  $\gamma_t$  does not intersect  $e$  we have that  $v_1$  and  $v_2$  belong to the same connected component of  $M \setminus \gamma_t$ . Hence,  $f_{t, v_1}^N = f_{t, v_2}^N$ .

Without loss of generality we may assume that the images of North and South poles under  $f$  are not vertices of  $T$ . Now we observe that  $d_v(0) = \deg(f) \neq 0$  and  $d_v(1) = 0$  for all vertices  $v$ .

The rest of the proof proceeds as in [FK]. Let  $V$  be the set of vertices of  $T$  and  $K(t) = \#\{v \in V : d_v = 0\}$ . We may perturb the homotopy slightly, so that  $\gamma_t$  passes through no more than one vertex for each  $t$ . Observation 1 implies that as  $t$  varies between 0 and 1  $K(t)$  will attain every value between 0 and  $2^N - 1$  (recall that  $N$  is the height of  $T$ ).

Consider what values  $K(t)$  can attain if  $\gamma_t$  intersects only one edge of  $T$ . Let  $v_1$  and  $v_2$  be two vertices of an edge  $e$  at distances (in the standard metric on the tree)  $i$  and  $i+1$ , respectively, from the root. By Observation 2 we have the following possibilities for the value of  $K(t)$ : the number of vertices in the connected component of  $M \setminus \gamma_t$  that contains  $v_1$  ( $2^N - 2^{N-i-1}$ ), the number of vertices in the connected component that contains  $v_2$  ( $2^{N-i-1} - 1$ ) or one of the min or max values (0 and  $2^N - 1$ ). Since this is true for every  $i$   $K(t)$  can attain at most  $2(N-1) + 2$  possible different values if  $\gamma_t$  intersects only one edge of  $T$ . Similarly, if  $\gamma_t$  has exactly  $j$  intersections then there are at most  $(2N)^j$  choices for the value of  $K(t)$ . If throughout the homotopy  $\gamma_t$  intersects at most  $k$  edges then  $K(t)$  attains no more than  $\sum_{j=1}^k (2N)^j \leq (2N)^{k+1}$  distinct values. Since all possible values are attained, we have  $(2N)^{k+1} \geq 2^N - 1$  and hence  $k \geq O(N/\log N)$ .

Since the distance between any two non-adjacent edges is greater than  $1/2$  we have that for some  $t$   $\gamma_t$  will be at least  $O(N/\log N)$  long.  $\square$

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